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Width of percolation transition in complex networks

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It is known that the critical probability for the percolation transition is not a sharp threshold. Actually it is a region of nonzero width Δp_c for systems of finite size. Here we present evidence that for complex networks $\Delta p_c \sim p_c/\ell$, where $\ell \sim N^{\nu_{opt}}$ is the average length of the percolation cluster, and N is the number of nodes in the network. For Erdős-Rényi graphs $\nu_{opt}=1/3$, while for scale-free networks with a degree distribution $P(k)\sim k^{-\lambda}$ and $3<\lambda<4$, $\nu_{opt}=(\lambda-3)/(\lambda-1)$. We show analytically and numerically that the *survivability* $S(p,\ell)$, which is the probability of a cluster to survive ℓ chemical shells at probability p, behaves near criticality as $S(p,\ell)=S(p_c,\ell)\exp[(p-p_c)\ell/p_c]$. Thus for probabilities inside the region $|p-p_c|<p_c/\ell$ the behavior of the system is indistinguishable from that of the critical point.

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I. INTRODUCTION

Recently the subject of networks has received much attention. It was realized that many real world systems, such as the Internet, can be successfully modeled as networks. Other examples include social networks such as the web of social contacts, and biological networks such as the protein interaction network and metabolic networks [1–3]. The problem of percolation on networks has also been studied extensively (e.g., [4]). Using percolation theory we can describe the resilience of the network to a breakdown of sites or links [5,6], epidemic spreading [7,8], and properties of optimal paths [9].

A typical percolation system consists of a d-dimensional grid of length L, in which the nodes or links are removed with some probability $1-p$, or are considered "conducting" with probability p (e.g., [10,11]). Below some critical probability p_c the system becomes disconnected into small clusters, i.e., it becomes impossible to cross from one side of the grid to the other by following the conducting links. Percolation is considered a geometrical phase transition exhibiting universality, critical exponents, upper critical dimension at $d=6$, etc. It was noted by Coniglio [12] that for systems of finite size L the transition from a connected to disconnected state has a width $\Delta p_c \sim L^{-1/\nu}$, where ν is a critical exponent related to the correlation length.

Percolation on networks was studied also from a mathematical viewpoint [4,13]. It was found that in Erdős-Rényi (ER) graphs with an average degree $\langle k\rangle$ the percolation threshold is $p_c=1/\langle k\rangle$. Below p_c the graph is composed of small clusters (most of them trees). As p approaches p_c trees of increasing order appear. At $p=p_c$ a giant component emerges and loops of all orders abruptly appear. However, for graphs of finite size N the percolation threshold has a finite width $\Delta p_c \sim N^{-1/3}$ [13], meaning that all attributes of criticality are present in the range $p \in [p_c-\Delta p_c, p_c+\Delta p_c]$. For example: the number of loops is negligible below $p_c+\Delta p_c$ [20].

In this paper we study the *survivability* of the network near the critical threshold. The survivability $S(p,\ell)$ is defined to be the probability of a connected cluster to "survive" up to ℓ chemical shells in a system with conductance probability p [14] (i.e, the probability that there exists at least one node at chemical distance ℓ from a randomly chosen node on the same cluster). At the critical point p_c, the survivability decays as a power law: $S(p_c,\ell)\sim \ell^{-x}$, where x is a universal exponent. Below p_c the survivability decays exponentially to zero, while above p_c it decays (exponentially) to a constant. Here we will derive analytically and numerically the functional form of the survivability above and below the critical point. We will show that near the critical point $S(p,\ell)=S(p_c,\ell)\exp[(p-p_c)\ell/p_c]$. Thus, given a system which has a maximal chemical length ℓ at the percolation threshold, for probabilities inside the range $|p-p_c|<p_c/\ell$ the behavior of the system is indistinguishable from that of the critical point. Hence we get $\Delta p_c \sim p_c/\ell$.

The maximal chemical length ℓ at the critical threshold, i.e., the length of the percolation cluster, was found to be $\ell \sim N^{\nu_{opt}}$ [9], where N is the number of nodes in the network. For Erdős-Rényi (ER) graphs $\nu_{opt}=1/3$, while for scale-free (SF) networks with a degree distribution $P(k)\sim k^{-\lambda}$ and $3<\lambda<4$, $\nu_{opt}=(\lambda-3)/(\lambda-1)$.

II. GENERAL FORMALISM

Consider a random graph with a degree distribution $P(k)$, i.e., a randomly chosen node has a probability $P(k)$ to have k links. The probability to reach a node of degree k by following a randomly chosen link is $P_1(k)=kP(k)/\langle k\rangle$ [15] where $\langle k\rangle$ is the average degree. Accordingly, we write the two corresponding probability generating functions (e.g., [8,15])

$$G_0(x)=\sum_{k=0}^{\infty} P(k)x^k \tag{1}$$

and

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$$G_1(x) = \frac{G_0'(x)}{G_0'(1)} = \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} kP(k)x^{k-1} = \sum_{k=1}^{\infty} P_1(k)x^{k-1}, \quad (2)$$

where $G_1(x)$ describes the probability that a node reached by following a random link has k outgoing links, not including the incoming link. For example, in ER graphs, $G_0(x) = G_1(x) = e^{\langle k \rangle (x-1)}$.

After randomly removing a fraction $1-p$ of the links (bond percolation), the probability for a randomly chosen node to have k remaining links in the diluted graph is given by [6]

$$\tilde{P}(k) = \sum_{k_0=k}^{\infty} P(k_0) \binom{k_0}{k} p^k (1-p)^{(k_0-k)}. \quad (3)$$

The corresponding probability generating functions $\tilde{G}_0(x) = \sum_{k=0}^{\infty} \tilde{P}(k)x^k$ and $\tilde{G}_1(x) = \sum_{k=1}^{\infty} \tilde{P}_1(k)x^{k-1}$ in the diluted graph are

$$\begin{aligned} \tilde{G}_0(x) &= \sum_{k=0}^{\infty} \left[\sum_{k_0=k}^{\infty} P(k_0) \binom{k_0}{k} p^k (1-p)^{(k_0-k)} \right] x^k \\ &= \sum_{k_0=0}^{\infty} P(k_0) \sum_{k=0}^{k_0} \binom{k_0}{k} (xp)^k (1-p)^{(k_0-k)} \\ &= \sum_{k_0=0}^{\infty} P(k_0)(1-p+px)^{k_0} = G_0(1-p+px), \end{aligned} \quad (4)$$

and [8,15]

$$\tilde{G}_1(x) = \frac{\tilde{G}_0'(x)}{\tilde{G}_0'(1)} = \frac{pG_0'(1-p+px)}{pG_0'(1)} = G_1(1-p+px). \quad (5)$$

For example, in ER graphs, $\tilde{G}_0(x) = \tilde{G}_1(x) = e^{\langle k \rangle ([1-p+px]-1)} = e^{\langle k \rangle p(x-1)}$.

We next define $M_\ell(x) = m_0 + m_1 x + m_2 x^2 + \cdots$ to be the generating function for the number of sites that exists on layer (i.e., chemical shell) ℓ starting from a *random node* on the diluted graph, and $N_\ell(x) = n_0 + n_1 x + n_2 x^2 + \cdots$ to be the corresponding function for the number of sites that exists on layer ℓ from a node reached by following a *random link*. In order to find $M_L(x)$ for some layer $L \gg 1$ we can write the following recursive relations [15,16]:

$$N_1(x) = \tilde{G}_1(x), \quad (6)$$

for $1 \leq \ell < L-1$,

$$N_{\ell+1}(x) = \tilde{G}_1(N_\ell(x)), \quad (7)$$

and similarly, for the final layer,

$$M_L(x) = \tilde{G}_0(N_{L-1}(x)). \quad (8)$$

Equation (7) means that the probability $n_i^{(\ell+1)}$ for reaching a branch having i nodes at layer $\ell+1$ is composed of the probability of reaching a node by following a link, and then reaching i nodes at layer ℓ by following all possible branches emerging from that node (see sketch in Fig. 1).

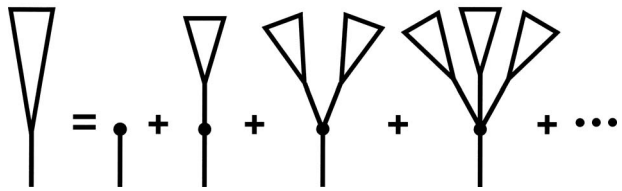

FIG. 1. A graphical sketch of the recursion relation (7). The probability to reach a branch having i nodes at layer $\ell+1$ may be represented as the sum of probabilities to reach a single node, to reach a node connected to a single branch having i nodes at layer ℓ, to reach a node connected to two branches having a total of i nodes at layer ℓ, etc. (see Sec. II).

As a simple demonstration, let us evaluate the probability $n_0^{(\ell+1)}$ to encounter zero nodes at layer $\ell+1$ of a branch. Taking the zeroth power in Eq. (7) we have $n_0^{(\ell+1)} = \tilde{P}_1(1) + \tilde{P}_1(2)n_0^{(\ell)} + \tilde{P}_1(3)[n_0^{(\ell)}]^2 + \cdots$, which means that the probability to reach zero nodes at layer $\ell+1$ (by following a link) is composed of the probability $\tilde{P}_1(1)$ to reach a node with no emerging branch, the probability $\tilde{P}_1(2)n_0^{(\ell)}$ to reach a node that has a single emerging branch with zero nodes at layer ℓ, the probability $\tilde{P}_1(3)[n_0^{(\ell)}]^2$ to reach a node having two branches such that both of them have zero nodes at layer ℓ, etc. (see Fig. 1). Similarly, Eq. (8) refers to $M_L(x)$, which gives the probability for the number of nodes at layer L reached by starting from a random node, rather than by following a random link [15].

Notice that $M_L(0) = m_0$ is the probability that there are zero nodes at layer L from a random node, i.e., the probability to die before layer L. Thus $\epsilon_L = 1 - M_L(0)$ is the probability to survive up to layer L. Similarly, $\epsilon_\ell = 1 - N_\ell(0)$ (where $1 \leq \ell \leq L-1$) is the probability for a branch to survive up to layer ℓ. From Eq. (7) we have

$$N_{\ell+1}(0) = \tilde{G}_1(N_\ell(0)) \quad (9)$$

$$1 - \epsilon_{\ell+1} = \tilde{G}_1(1-\epsilon_\ell) = G_1(1-p+p[1-\epsilon_\ell]) \quad (10)$$

Thus for $1 \leq \ell < L-1$,

$$\epsilon_{\ell+1} = 1 - G_1(1-p\epsilon_\ell), \quad (11)$$

and for the final layer L, we have [Eq. (8)],

$$\epsilon_L = 1 - G_0(1-p\epsilon_{L-1}), \quad (12)$$

which gives the survivability at layer L [16].

III. ERDŐS-RÉNYI GRAPHS

For ER graphs, $G_0(x) = G_1(x) = e^{\langle k \rangle (x-1)}$ and Eq. (11) gives

$$\begin{aligned} \epsilon_{\ell+1} &= 1 - e^{\langle k \rangle ([1-p\epsilon_\ell]-1)} = 1 - e^{-p\langle k \rangle \epsilon_\ell} \\ &= 1 - \left[1 - p\langle k \rangle \epsilon_\ell + \frac{p^2 \langle k \rangle^2}{2} \epsilon_\ell^2 - \cdots \right] \\ &= \frac{p}{p_c} \epsilon_\ell - \frac{p^2 \langle k \rangle^2}{2} \epsilon_\ell^2 + \cdots, \end{aligned} \quad (13)$$

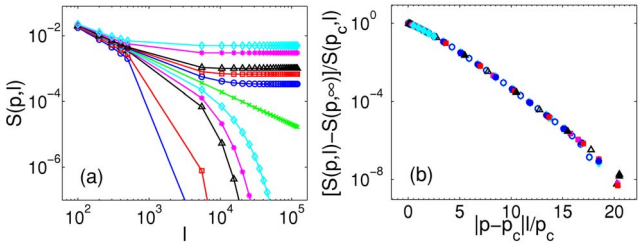

FIG. 2. (Color online) (a) The survivability $S(p,\ell)$ for an ER graph with $\langle k\rangle=5$, numerically calculated for different values of p: p_c, $p_c\pm5\times10^{-4}$, $p_c\pm3\times10^{-4}$, $p_c\pm1\times10^{-4}$, $p_c\pm6.66\times10^{-5}$, and $p_c\pm3.33\times10^{-5}$. For $p=p_c$ the survivability decays to zero according to a power law: $S(p_c,\ell)\sim\ell^{-1}$. For $p<p_c$, $S(p,\ell)\to0$, while for $p>p_c$, $S(p,\ell)\to$const. The decay is exponential (to zero or to a constant) according to Eqs. (16) and (17). (b) Scaling of the survivability for different values of p, ℓ, and $\langle k\rangle$. Shown is $[S(p,\ell)-S(p,\infty)]/S(p_c,\ell)$ vs $|p-p_c|\ell/p_c$ for ER graphs with $\langle k\rangle=5$ (unfilled symbols) and $\langle k\rangle=10$ (filled symbols). The collapse of all curves on an exponential function (for large ℓ) supports the scaling relations (16) and (17).

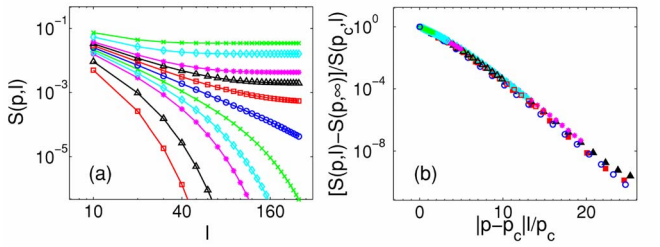

FIG. 3. (Color online) (a) The survivability $S(p,\ell)$ for a SF network with $\lambda=3.5$, numerically calculated for different values of p: p_c, $p_c\pm6\times10^{-2}$, $p_c\pm4\times10^{-2}$, $p_c\pm2\times10^{-2}$, $p_c\pm1.33\times10^{-2}$, and $p_c\pm6.66\times10^{-3}$. For $p=p_c$ the survivability decays to zero according to a power law: $S(p_c,\ell)\sim\ell^{-2}$. For $p\neq p_c$, $S(p,\ell)$ decays exponentially (to zero or to a constant) according to Eqs. (16) and (17). (b) Scaling of the survivability for different values of p, ℓ, and λ. Shown is $[S(p,\ell)-S(p,\infty)]/S(p_c,\ell)$ vs $|p-p_c|\ell/p_c$ for SF graphs with $\lambda=3.5$ (filled symbols) and $\lambda=5$ (unfilled symbols). For all cases $m=2$. Due to numerical difficulties only curves with $p<p_c$ are shown.

where $p_c=1/\langle k\rangle$. Substituting $\delta\equiv p-p_c$, we get

$$\epsilon_{\ell+1}=\frac{p_c+\delta}{p_c}\epsilon_\ell-(p_c+\delta)^2\frac{\langle k\rangle^2}{2}\epsilon_\ell^2+\cdots\approx\epsilon_\ell+\frac{\delta}{p_c}\epsilon_\ell-\frac{1}{2}\epsilon_\ell^2, \tag{14}$$

where we have left only terms of second order in ϵ_ℓ, δ [21]. We thus get

$$\frac{d\epsilon_\ell}{d\ell}\approx\epsilon_{\ell+1}-\epsilon_\ell=-\frac{1}{2}\epsilon_\ell^2+\frac{\delta}{p_c}\epsilon_\ell. \tag{15}$$

At criticality, $\delta=0$ and the solution to this equation is $\epsilon_\ell\sim\ell^{-1}$. The additional term suggests the following solution near criticality $\epsilon_\ell\sim\ell^{-1}\exp(\delta\ell/p_c)$. Note that for ER graphs Eqs. (11) and (12) are the same, and thus the survivability ϵ_L at the final iteration also has the same form $\epsilon_L\sim L^{-1}\exp(\delta L/p_c)$. The above result can be written as

$$S(p,\ell)=S(p_c,\ell)\exp\left(\frac{1}{p_c}(p-p_c)\ell\right). \tag{16}$$

In order to check this result we numerically calculated the survivability $S(p,\ell)$ near p_c according to the recursive relations (11) and (12).

Figure 2(a) shows the survivability $S(p,\ell)$ for different values of p. For $p=p_c$ the survivability decays as a power law, while above and below there is an exponential decay, either to zero (for $p<p_c$) or to a constant (for $p>p_c$). Figure 2(b) shows that all curves of the survivability $S(p,\ell)$ from (a) can be rescaled such that they all collapse. Moreover, scaled survivabilities from all different graphs with different values of $\langle k\rangle$ (i.e., different values of p_c) also collapse on the same curve. However, Eq. (16) is true only below the percolation threshold where there is no giant component. Above the percolation threshold there is an exponential decay to a nonzero constant, and the generalized expression is

$$S(p,\ell)=S(p_c,\ell)\exp\left(-\frac{1}{p_c}|p-p_c|\ell\right)+P_\infty, \tag{17}$$

where P_∞ is the probability for a randomly chosen site to be inside the percolation cluster [22]. Indeed, setting $\epsilon_{\ell+1}=\epsilon_\ell$ in the recursive relation $\epsilon_{\ell+1}=1-e^{-p\langle k\rangle\epsilon_\ell}$, the resulting "steady state" solution is $\epsilon_\ell=P_\infty$ [13].

IV. SCALE-FREE GRAPHS

Scale-free graphs can be taken to have a degree distribution of the form $P(k)=ck^{-\lambda}$, where $c\approx(\lambda-1)m^{\lambda-1}$ and m is the minimal degree [6]. In order to solve Eq. (11) we have to evaluate

$$G_1(1-p\epsilon_\ell)=\frac{1}{\langle k\rangle}\sum kP(k)(1-p\epsilon_\ell)^{k-1}. \tag{18}$$

Expanding by powers of ϵ, and inserting $P(k)=ck^{-\lambda}$ with $3<\lambda<4$, we get [17,18,23]

$$\sum kP(k)(1-\epsilon)^{k-1}\approx\langle k\rangle-\langle k(k-1)\rangle\epsilon+\frac{c}{2}\Gamma(4-\lambda)\epsilon^{\lambda-2}. \tag{19}$$

Thus Eq. (11) becomes

$$\epsilon_{\ell+1}\approx1-\frac{1}{\langle k\rangle}\left[\langle k\rangle-\langle k(k-1)\rangle p\epsilon_\ell+\frac{c}{2}\Gamma(4-\lambda)(p\epsilon_\ell)^{\lambda-2}\right]$$
$$=\frac{p}{p_c}\epsilon_\ell-\frac{c}{2\langle k\rangle}\Gamma(4-\lambda)p^{\lambda-2}\epsilon_\ell^{\lambda-2}, \tag{20}$$

where $p_c=\langle k\rangle/\langle k(k-1)\rangle$ [6]. Taking $p=p_c+\delta$, and substituting $A\equiv(c/2\langle k\rangle)\Gamma(4-\lambda)p_c^{\lambda-2}$, we get

$$\epsilon_{\ell+1} = \frac{p_c+\delta}{p_c}\epsilon_\ell - \frac{c}{2\langle k\rangle}\Gamma(4-\lambda)(p_c+\delta)^{\lambda-2}\epsilon_\ell^{\lambda-2}$$
$$= \epsilon_\ell + \frac{\delta}{p_c}\epsilon_\ell - A\left(1+\frac{\delta}{p_c}\right)^{\lambda-2}\epsilon_\ell^{\lambda-2}$$
$$\approx \epsilon_\ell - A\epsilon_\ell^{\lambda-2} + \frac{\delta}{p_c}[\epsilon_\ell - A(\lambda-2)\epsilon_\ell^{\lambda-2}] \quad (21)$$

For large ℓ, $\epsilon_\ell \ll 1$. Taking into account that $\lambda-2>1$ we have $\epsilon_\ell^{\lambda-2} \ll \epsilon_\ell$. Therefore,

$$\frac{d\epsilon_\ell}{d\ell} \approx \epsilon_{\ell+1} - \epsilon_\ell = -A\epsilon_\ell^{\lambda-2} + \frac{\delta}{p_c}\epsilon_\ell. \quad (22)$$

For $\delta=0$ the solution is $\epsilon_\ell \sim \ell^{-x}$ with $x=1/(\lambda-3)$. The additional term suggests the following solution near criticality, $\epsilon_\ell \sim \ell^{-x}\exp(\delta\ell/p_c)$. The last iteration [Eq. (12)] can be shown to give the same behavior for ϵ_L. A similar form can be found also for $\lambda>4$ [24]. The scaling form for SF networks is confirmed by numerical simulations as shown in Figs. 3(a) and 3(b).

V. SUMMARY AND CONCLUSIONS

We have shown analytically and numerically that the survivability in ER and SF graphs scales according to Eqs. (16) and (17) near the critical point. Thus, the scaling form of the survivability near the critical probability obeys the following scaling relation (for $p<p_c$)

$$S(p,\ell) = S(p_c,\ell)\exp\left(\frac{p-p_c}{\Delta p_c}\right), \quad (23)$$

where $\Delta p_c = p_c/\ell$. Given a system with a maximal chemical length ℓ at criticality, for all values of conductivity p inside the range $[p_c-\Delta p_c, p_c+\Delta p_c]$, the survivability behaves similar to the power law $S(p_c,\ell)\sim\ell^{-x}$ found at $p=p_c$. Thus, the width of the critical threshold is $\Delta p_c = p_c/\ell$, where ℓ is the chemical length of the percolation cluster. For ER graphs, $\ell \sim N^{1/3}$, while for SF networks with $3<\lambda<4$, $\ell \sim N^{(\lambda-3)/(\lambda-1)}$.

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[1] A.-L. Barabási, *Linked: The New Science of Networks* (Perseus Books Group, Cambridge, MA 2002).
[2] S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks* (Oxford University Press, Oxford, 2003).
[3] R. Pastor-Satorras and A. Vespignani, *Evolution and Structure of the Internet* (Cambridge University Press, Cambridge, 2004).
[4] R. Albert *et al.*, Rev. Mod. Phys. **74**, 47 (2002).
[5] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. **85**, 5468 (2000).
[6] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000).
[7] R. Cohen, S. Havlin, and Daniel ben-Avraham, Phys. Rev. Lett. **91**, 247901 (2003).
[8] M. E. J. Newman, Phys. Rev. Lett. **95**, 108701 (2005).
[9] L. A. Braunstein, S. V. Buldyrev, R. Cohen, S. Havlin, and H. E. Stanley, Phys. Rev. Lett. **91**, 168701 (2003).
[10] A. Bunde and S. Havlin, *Fractals and Disordered Systems* (Springer, New York, 1996).
[11] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor and Francis, London, 1992).
[12] A. Coniglio, J. Phys. A **15**, 3829 (1982).
[13] P. Erdös and A. Rényi, Publ. Math., Inst. Hungar. Acad. Sci. **5**, 17 (1960); B. Bollobás, *Random Graphs* (Cambridge University Press, Cambridge, 2001).
[14] F. Tzschichholz, A. Bunde, and S. Havlin, Phys. Rev. A **39**, 5470 (1989).
[15] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).
[16] L. A. Braunstein *et al.*, in *Lecture Notes in Physics* (Springer, Berlin, 2004), Vol. 650, p. 127.
[17] R. Cohen and S. Havlin, Physica A **336**, 6 (2004).
[18] R. Cohen and S. Havlin, *Complex Networks* (Cambridge University Press, Cambridge, in press).
[19] R. Cohen, D. ben-Avraham, and S. Havlin, Phys. Rev. E **66**, 036113 (2002).
[20] O. Riordan and P. L. Krapivsky (private communication).
[21] We assume that $p<p_c$ and thus $\epsilon_\ell \ll 1$ for large ℓ.
[22] $S(p,\ell\to\infty)$ is the probability that if we start from a randomly chosen site, we will survive an infinite chemical distance. This equals the probability P_∞ that the chosen site resides in the giant component. P_∞ obeys the transcendental equation $P_\infty = 1-e^{-\langle k\rangle p P_\infty}$.
[23] We expand up to second order in ϵ. The last term is the remainder of the series expansion, $R_2 \approx \frac{1}{2}\epsilon^2 \Sigma k(k-1)(k-2)ck^{-\lambda}(1-\epsilon)^{k-3} \approx \frac{1}{2}\epsilon^2 ce^{3\epsilon}\int k^{3-\lambda}e^{-\epsilon k}dk \approx (c/2)\Gamma(4-\lambda)\epsilon^{\lambda-2}$, which arises from the fact that the second derivative diverges (see [18] for details).
[24] In this range the behavior is similar to ER graphs [19].